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Heat transfer in a periodic boundary layer near a two-dimensional stagnation point

By HIROSHI ISHIGAKI
National Aerospace Laboratory, Kakuda Branch, Miyagi, Japan

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Following the previous velocity-field study (Ishigaki 1970), this paper studies how the temperature field in the laminar boundary layer near a two-dimensional stagnation point responds to the main-stream oscillation. The time-mean temperature field is of particular interest and is studied in detail. The velocity field is treated as known and is taken from the previous paper. In § 3 the solutions over the whole frequency range are obtained under the assumption of small amplitude oscillation and the results are compared with the existing approximate solutions for low and high frequency in terms of heat transfer. Time-mean heat transfer decreases at low frequency, but slightly increases at high frequency. Two factors that cause time-mean modification of the temperature field are examined quantitatively. In § 4 the finite amplitude case is treated under the assumption of high-frequency oscillation and a few examples of the time-mean temperature profile are shown.

1. Introduction

In a previous paper Ishigaki (1970) studied the time-mean characteristics of the laminar boundary layer near a two-dimensional stagnation point for the case when the velocity of the oncoming stream relative to the body oscillates. In this paper the corresponding heat-transfer problem is studied for the case when the body is heated to a constant temperature, and it is shown how the temperature field responds to the main-stream oscillation.

Theoretical studies on heat-transfer fluctuation in a periodic boundary layer near a stagnation point have been made by Lighthill (1954), Gribben (1961) and Mori & Tokuda (1966) for two-dimensional flow and by Yeh & Yang (1965) and Ghohal & Ghohal (1970) for three-dimensional flow. These studies show that, at constant velocity amplitude of the main-stream oscillation, the amplitude of the heat-transfer fluctuation decreases with increasing frequency and the phase always lags that of the main-stream oscillation, the amplitude being inversely proportional to the frequency and the phase lag approaching the limit 90° at high frequencies. The time-mean heat transfer in the two-dimensional stagnation flow was studied by Gersten (1965), who gave the result that the time-mean heat-transfer rate is smaller than that without oscillation.

In these studies approximate solutions were obtained for the cases of low and high frequency; thus the uncertainty at intermediate frequency was inevitable.

Moreover, these studies were restricted to small amplitude oscillation. In § 3 of the present paper the solutions over the whole frequency range are obtained under the assumption of small amplitude oscillation, and the treatment for finite amplitude oscillation is made in § 4 under the assumption of high-frequency oscillation.

There is a time-mean secondary flow induced by the oscillation, whose origin lies in the nonlinearity of the governing equation. This effect was first shown clearly by Schlichting (1932) in the case of a circular cylinder oscillating in a fluid at rest. When a mean flow is present, the time-mean velocity profile differs from that without oscillation because of the appearance of the secondary flow. Lin (1957) showed that the difference between them is strongly influenced by a pressure gradient for high-frequency oscillation, and a quantitative comparison of the effect of the pressure gradient on the skin friction was given by Ishigaki (1970, 1971*a*) for Hiemenz and Blasius flow. This secondary flow naturally affects the time-mean heat transfer.

In an incompressible fluid flow with negligible viscous dissipation the oscillation introduces a time-mean modification in the temperature field through two effects. The first is the heat convection by the above-mentioned secondary flow. The second effect arises from the correlation between the fluctuations of velocity and temperature, and, as in the heat-transfer analysis of the turbulent flow, this effect may conveniently be treated as either an apparent heat source or an apparent change in heat conduction. It is shown in § 3 how the time-mean heat transfer is influenced by these two factors, and some remarks concerning the effect of pressure gradient on heat transfer are given in § 5.

2. Basic equations

Let us consider a two-dimensional unsteady laminar flow with negligible viscous dissipation of an incompressible fluid with constant properties. The boundary-layer equations of continuity, momentum and energy are written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where x and y are distances parallel and normal to the surface, u and v are the corresponding velocity components, t is time, T temperature, U the velocity at the outer edge of the boundary layer, ν kinematic viscosity and κ thermal conductivity. The boundary conditions are

$$u=v=0, \quad T=T_w \quad \text{at} \quad y=0, \quad u=U(x,t), \quad T=T_\infty \quad \text{as} \quad y\to\infty, \tag{4}$$

where T_w and T_∞ are constants. Near a front stagnation point of a cylindrical body, the main-stream velocity is assumed to be the real part of the form

$$U(x,t)=Ax(1+\epsilon\, e^{i\omega t}), \tag{5}$$

where ω is frequency and A and ϵ are constants. Furthermore, the non-dimensional temperature is defined as

$$\theta = (T - T_\infty)/(T_w - T_\infty). \tag{6}$$

Since the temperature field does not affect the velocity field because of the constant-properties assumption, the particular velocity field is treated as known, being taken from Ishigaki (1970), where it was calculated.

3. Small amplitude case

When ϵ is small, the solutions are assumed in the forms

$$\left.\begin{aligned} u(x,y,t) &= Ax[f'(\eta)+\epsilon g'(\eta)\,e^{i\omega t}+\epsilon^2\{G'(\eta)+G_a'(\eta)\,e^{2i\omega t}\}+\ldots],\\ v(x,y,t) &= -(A\nu)^{\frac{1}{2}}\,[f(\eta)+\epsilon g(\eta)\,e^{i\omega t}+\epsilon^2\{G(\eta)+G_a(\eta)\,e^{2i\omega t}\}+\ldots],\\ \theta(x,y,t) &= h(\eta)+\epsilon k(\eta)\,e^{i\omega t}+\epsilon^2\{K(\eta)+K_a(\eta)\,e^{2i\omega t}\}+\ldots, \end{aligned}\right\} \tag{7}$$

in which only the real parts are to be taken and primes denote differentiation with respect to $\eta = y(A/\nu)^{\frac{1}{2}}$. The continuity equation (1) is then satisfied identically. By substituting (5), (6) and (7) into (2) and (3) and equating terms of the same order in ϵ, sets of ordinary differential equations are obtained. The functions $f(\eta)$ and $h(\eta)$ are the well-known steady-flow solutions (see Schlichting 1968).

The equation for $k(\eta)$ is

$$(1/Pr)\,k''+fk'-i\sigma k = -gh', \tag{8}$$

with $\qquad k = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \to \infty,$

where $Pr = \nu/\kappa$ is the Prandtl number and $\sigma = \omega/A$ is the frequency parameter. This equation is solved numerically for $Pr = 0{\cdot}72$ and various values of σ. Approximate solutions have been obtained for the cases when σ is small and large. Mori & Tokuda (1966) obtained the following approximate formulae for the heat-transfer fluctuation:

$$k'(0)/h'(0) = \begin{cases} \frac{1}{2} - 0{\cdot}349 i\sigma + 0{\cdot}229(i\sigma)^2 + \ldots & \text{(small } \sigma\text{)}, \quad (9a)\\ \dfrac{0{\cdot}541}{i\sigma} + \dfrac{1{\cdot}082}{(i\sigma)^2} - \dfrac{4{\cdot}892}{(i\sigma)^2\,(i\sigma)^{\frac{1}{2}}} + \ldots & \text{(large } \sigma\text{)}. \quad (9b)\end{cases}$$

The amplitude and phase angle of heat-transfer fluctuations of order ϵ are shown in figure 1, in which present numerical results are shown by solid lines and the above approximate results are shown by dotted lines, broken lines showing the asymptotic values, the first term only in (9*b*), obtained by Lighthill. Mori & Tokuda also presented the approximating function obtained by joining (9*a*) and (9*b*) smoothly, and the values estimated from the function lie within 1·5% of the present results.

The equation for the time-independent function $K(\eta)$ is

$$(1/Pr)\,K''+fK' = -Gh' - \tfrac{1}{2}(g_r k_r' + g_i k_i'), \tag{10}$$

with $\qquad K = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \to \infty,$

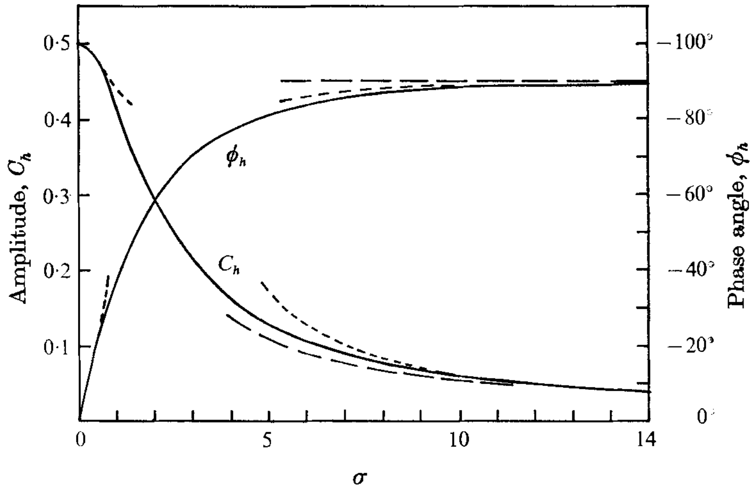

FIGURE 1. Amplitude and phase angle of heat-transfer fluctuations for $Pr = 0{\cdot}72$. - - - -, Mori & Tokuda (1966); — — —, Lighthill (1954).

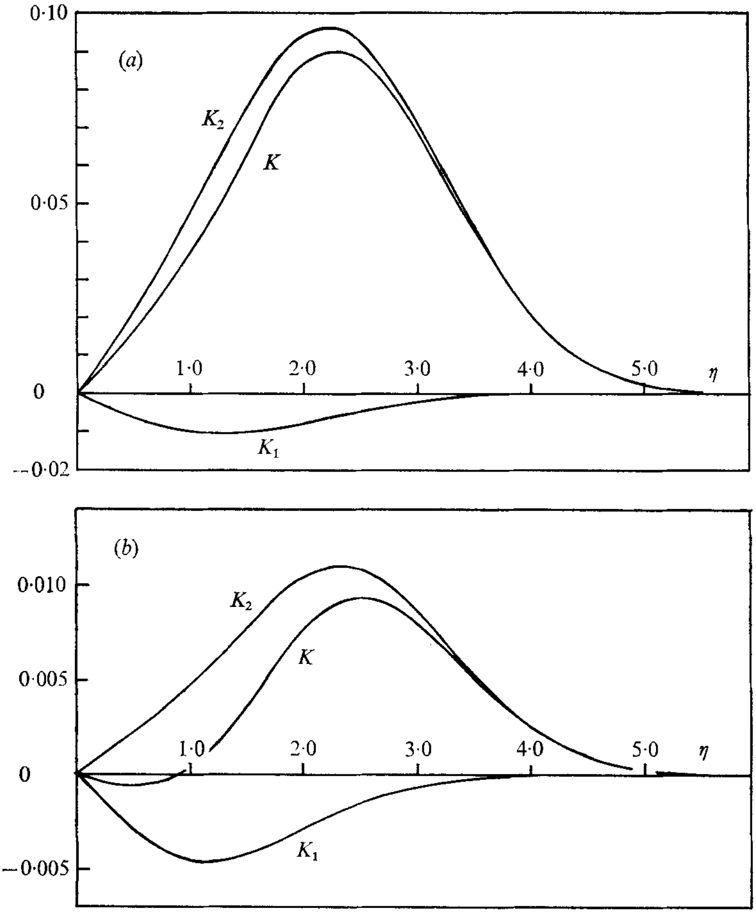

FIGURE 2. Plots of K, K_1 and K_2 for $Pr = 0{\cdot}72$. (a) $\sigma = 0$. (b) $\sigma = 5{\cdot}0$.

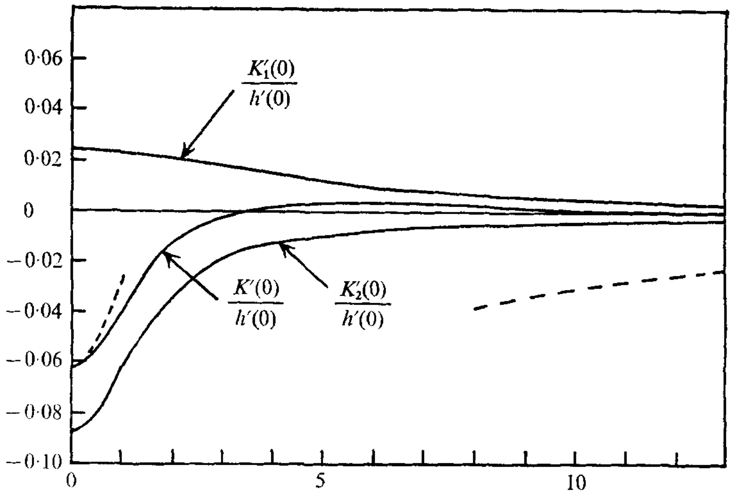

FIGURE 3. Plots of $K'(0)$, $K_1'(0)$ and $K_2'(0)$ for $Pr = 0{\cdot}72$. - - -, Gersten (1965).

where subscripts r and i denote real and imaginary parts. It can be seen that the time-mean deviation of the temperature field from that without oscillation arises from two factors. To see the contributions of each factor, we let

$$K(\eta) = K_1(\eta) + K_2(\eta), \tag{11}$$

which then satisfy the following equations:

$$(1/Pr)\,K_1'' + fK_1' = -Gh', \tag{12a}$$

$$(1/Pr)\,K_2'' + fK_2' = -\tfrac{1}{2}(g_r k_r' + g_i k_i'), \tag{12b}$$

$$K_1 = K_2 = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \to \infty.$$

K_1 is associated with the heat convection by the secondary flow and K_2 is associated with the apparent heat source due to the correlation between fluctuations of velocity and temperature. These equations are solved numerically for $Pr = 0{\cdot}72$. K, which is proportional to the time-mean modification of the temperature field, is shown in figure 2(*a*) for $\sigma = 0$ and in figure 2(*b*) for $\sigma = 5$, the functions K_1 and K_2 also being shown in these figures. Figure 3 shows the variation of $K'(0)$, $K_1'(0)$ and $K_2'(0)$ with σ. The time-mean heat flux

$$\bar{q} = -\frac{1}{2\pi}\int_0^{2\pi} \lambda \left(\frac{\partial T}{\partial y}\right)_{y=0} dt$$

is given to be

$$\frac{\bar{q}}{q_0} = 1 + \epsilon^2 \frac{K'(0)}{h'(0)}, \tag{13}$$

in which $q_0 = 0{\cdot}5014\lambda(T_w - T_\infty)(A/\nu)^{\frac{1}{2}}$ is the heat flux without oscillation and λ is the heat conductivity of the fluid. Dotted lines in figure 3 show the following approximate results for $Pr = 0{\cdot}72$ obtained by the same method as that employed by Gersten (1965):

$$K'(0)/h'(0) = \begin{cases} -\frac{1}{16} + 0{\cdot}034\sigma^2 + \dots \quad (\text{small } \sigma), & (14a) \\ -0{\cdot}307/\sigma + \dots \quad (\text{large } \sigma). & (14b) \end{cases}$$

For $Pr = 0{\cdot}70$ Gersten gave the coefficient $0{\cdot}031$ instead of $0{\cdot}034$ and $-0{\cdot}302$ for $-0{\cdot}307$. The function K_1 acts to promote heat transfer while K_2 prevents heat transfer. Figure 2(*a*) shows that the temperature rise due to the apparent heat source overcomes the temperature decrease due to the convection by the secondary flow, thus the net heat transfer decreases at low frequency. At some intermediate frequency they may cancel out near the wall to produce no net heat-transfer deviation. At high frequency, as is seen from figure 2(*b*), K_1 overcomes K_2 near the wall and the net heat transfer slightly increases. At higher frequencies both K_1 and K_2 decrease in magnitude and the effect of oscillation on the time-mean heat transfer tends to zero. Figure 3 clearly shows the dependence of heat-transfer variation on σ.

4. Finite amplitude case

In the following it is intended to apply Lin's (1957) method to the heat-transfer problem. As in turbulent-flow analysis, it is assumed that the velocity and temperature in the boundary layer are separated into time-mean and fluctuating components:

$$\left.\begin{aligned} u(x,y,t) &= \overline{u}(x,y) + u_t(x,y,t), \\ v(x,y,t) &= \overline{v}(x,y) + v_t(x,y,t), \\ \theta(x,y,t) &= \overline{\theta}(x,y) + \theta_t(x,y,t), \\ \overline{u_t} = \overline{v_t} &= \overline{\theta_t} = 0, \end{aligned}\right\} \tag{15}$$

where an overbar denotes a time-mean quantity. On substituting (15) into (1), (2) and (3) and taking time average, the time-mean equations are obtained. The fluctuating equations are obtained by subtracting these time-mean equations from the corresponding full equations. The time-mean and fluctuating equations for the temperature field are

$$\overline{u}\frac{\partial\overline{\theta}}{\partial x} + \overline{v}\frac{\partial\overline{\theta}}{\partial y} + \overline{u_t\frac{\partial\theta_t}{\partial x}} + \overline{v_t\frac{\partial\theta_t}{\partial y}} = \kappa\frac{\partial^2\overline{\theta}}{\partial y^2}, \tag{16}$$

$$\frac{\partial\theta_t}{\partial t} + \overline{u}\frac{\partial\theta_t}{\partial x} + u_t\frac{\partial\overline{\theta}}{\partial x} + u_t\frac{\partial\theta_t}{\partial x} - \overline{u_t\frac{\partial\theta_t}{\partial x}} + \overline{v}\frac{\partial\theta_t}{\partial y} + v_t\frac{\partial\overline{\theta}}{\partial y} + v_t\frac{\partial\theta_t}{\partial y} - \overline{v_t\frac{\partial\theta_t}{\partial y}} = \kappa\frac{\partial^2\theta_t}{\partial y^2}, \tag{17}$$

$$\overline{\theta} = 1, \quad \theta_t = 0 \quad \text{at} \quad y = 0, \qquad \overline{\theta} = \theta_t = 0 \quad \text{as} \quad y \to \infty.$$

For high-frequency oscillation the velocity field is given by

$$\left.\begin{aligned} u_t &= \epsilon Ax\{1 - \exp[-(i\omega/\nu)^{\frac{1}{2}}y]\}e^{i\omega t}, \\ v_t &= -\epsilon A(\nu/i\omega)^{\frac{1}{2}}\{y(\nu/i\omega)^{\frac{1}{2}} - 1 + \exp[-(i\omega/\nu)^{\frac{1}{2}}y]\}e^{i\omega t}, \end{aligned}\right\} \tag{18}$$

for fluctuating components (Lin 1957), and

$$\overline{u} = AxF'(\eta), \quad \overline{v} = -(A\nu)^{\frac{1}{2}}F(\eta), \tag{19}$$

for time-mean components (Ishigaki 1970).

When the flow oscillation is of high frequency, (17) may be simplified to

$$\frac{\partial\theta_t}{\partial t} + u_t\frac{\partial\overline{\theta}}{\partial x} + v_t\frac{\partial\overline{\theta}}{\partial y} = \kappa\frac{\partial^2\theta_t}{\partial y^2}. \tag{20}$$

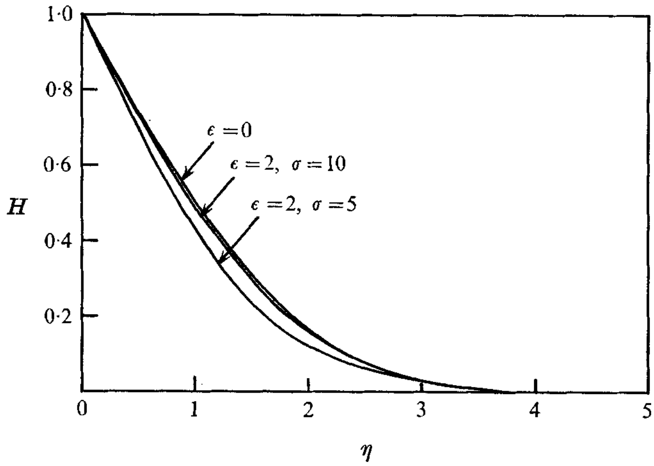

FIGURE 4. Time-mean temperature profiles for $Pr = 0{\cdot}72$ and $\epsilon = 0$ ($H'(0) = -0{\cdot}5014$), $\epsilon = 2, \sigma = 10$ ($H'(0) = -0{\cdot}5299$), and $\epsilon = 2, \sigma = 5$ ($H'(0) = -0{\cdot}6059$).

This is a reasonable simplification when the following assumptions are valid:

$$\frac{\partial \theta_t}{\partial t} \gg \bar{u}\frac{\partial \theta_t}{\partial x}, \quad u_t \frac{\partial \theta_t}{\partial x}, \quad \text{or} \quad \sigma \gg 1, \epsilon. \tag{21}$$

These are equivalent to the assumptions made in the velocity-field analysis. If we write

$$\theta_t(x,y,t) = p(\eta)\, e^{i\omega t}, \quad \bar{\theta}(x,y) = H(\eta), \tag{22}$$

the fluctuating temperature equation (20) becomes

$$\frac{1}{Pr}p'' - i\sigma p = -\epsilon H' \left\{\eta - \frac{1}{(i\sigma)^{\frac{1}{2}}} + \frac{1}{(i\sigma)^{\frac{1}{2}}}\exp\left[-(i\sigma)^{\frac{1}{2}}\eta\right]\right\}, \tag{23}$$

with $\qquad p = 0 \quad \text{at} \quad \eta = 0 \quad \text{and as} \quad \eta \to \infty.$

The formal solution which contains the unknown function $H(\eta)$ is obtained as

$$p(\eta) = \frac{\epsilon H'}{i\sigma}\left\{\eta - \frac{1}{(i\sigma)^{\frac{1}{2}}} - \frac{Pr}{(1-Pr)(i\sigma)^{\frac{1}{2}}}\exp\left(-(i\sigma)^{\frac{1}{2}}\eta\right)\right\} + \frac{\epsilon H'(0)}{(1-Pr)\, i\sigma\, (i\sigma)^{\frac{1}{2}}}\exp\left[-(iPr\,\sigma)^{\frac{1}{2}}\eta\right]. \tag{24}$$

If we replace $H(\eta)$ by the known function $h(\eta)$ in (7), this solution reduces to Lighthill's asymptotic form of the small amplitude case. Substituting (18), (19) and (24) into (16), the time-mean temperature equation becomes

$$\frac{1}{Pr}H'' + FH' = \frac{\epsilon^2\eta}{2(1-Pr)\,\sigma}\left\{Pr\,H'\sin\left[\left(\frac{\sigma}{2}\right)^{\frac{1}{2}}\eta\right]\exp\left[-\left(\frac{\sigma}{2}\right)^{\frac{1}{2}}\eta\right] - Pr^{\frac{1}{2}}H'(0)\sin\left[\left(\frac{Pr\,\sigma}{2}\right)^{\frac{1}{2}}\eta\right]\exp\left[-\left(\frac{Pr\,\sigma}{2}\right)^{\frac{1}{2}}\eta\right]\right\}, \tag{25}$$

$$H = 1 \quad \text{at} \quad \eta = 0, \qquad H = 0 \quad \text{as} \quad \eta \to \infty.$$

This becomes a more suitable form to calculate when both sides are divided by $H'(0)$. Solutions are obtained numerically for given values of ϵ and σ. Some typical examples of temperature profiles in the boundary layer are shown in figure 4. This figure shows that, even if the oscillation amplitude becomes large, the time-mean temperature field may be influenced by high-frequency oscillation only weakly. Thus, the function $H(\eta)$ in (24) can be approximately replaced by $h(\eta)$, and Lighthill's asymptotic form is still valid.

5. Concluding remarks

Concerning the heat-transfer fluctuation, the general trends mentioned in the introduction to this paper are confirmed by more exact results in § 3 and the features at high frequency are proved to be retained for finite amplitude oscillation in § 4. The time-mean temperature field is the main interest of this paper and is studied in detail. The time-mean heat transfer decreases at low frequency but slightly increases at high frequency, the effect of oscillation tending to zero at higher frequencies. Two factors that cause the time-mean deviation of the temperature field from that without oscillation are examined and the above variation of heat transfer with frequency is explained.

In Blasius flow in which pressure gradient is zero, negligible secondary flow at high frequency is expected from Lin's theory and K_1 will be very small. Since K_2 is not influenced by the pressure gradient to first order, the time-mean heat-transfer increase cannot be expected at any frequency in Blasius flow (see Ishigaki 1971*b*). Both the velocity and temperature fields are hardly influenced by the pressure gradient at low frequency. As the heat-transfer increase in stagnation flow at high frequency is unexpectedly slight, however, it may be concluded that the practical effect of the pressure gradient on periodic-boundary-layer heat transfer is negligibly small, although the effect on the velocity field is strong. In some literature discussing the effect of main-stream turbulence on laminar heat transfer, it has sometimes been said that the considerable effect near a stagnation point of a cylindrical body might be explained qualitatively by Lin's theory (e.g. Schlichting 1968). The experimental results on the main-stream turbulence effect show the two remarkable features that negligible heat-transfer effect is observed in Blasius flow and skin frictions in Blasius and Hiemenz flow are not subjected to a marked influence. Considering the results of the periodic-flow heat-transfer studies collectively, the above two features cannot be explained by the similarity with the effect of oscillation and therefore the above hypothesis may not be accepted.

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